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Gauge-Variance of the Dirac Bracket

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Abstract

This work is devoted to the study of the change in the functional form of the minus Dirac bracket under pure gauge transformations (in the sense of Levy–Leblond). We found a closed formula which expresses this change and we use it to discuss the relevance of a gauge transformation for the skew-symmetric (Bose-like) quantization procedure of constrained classical models. We found necessary conditions which are to be fulfilled if the gauge transformation is to induce a mere change of representation at the quantum level. It is shown, by considering a simple example, that these conditions can be violated. We conclude then that adding a total time derivative to the Lagrangian of a classical model can drastically change the physical properties of the quantized Bose-like counterpart. A similar result has been detected previously for two particular systems quantized through the symmetric (Fermi-like) rule of quantization.

1. *Introduction*

For some important classical problems the ordinary canonical formalism fails because the canonical variables satisfy a certain number of constraints. A detailed study of the general features of these abnormal systems (called in the following constrained systems) and of the skew-symmetric (Bose-like) quantization rule associated to them has been given by Dirac (1950, 1951, 1958, 1964).† Dirac's procedure has been extended posteriorly by Franke and Kálnay (1970) to cover the symmetric (Fermi-like) quantization problem of constrained classical models. In both cases the most interesting situation arises when an irreducible set of minus (respectively plus) second-class constraints exists: $\{\theta_-\}$ (resp. $\{\theta_+\}$), These two are different subsets

† Dirac's theory will be briefly reviewed below.

of the whole set of constraints the problem at hand may have and are characterized by the fact that the matrices whose elements are the brackets $\{\theta_{\mp}{}^{a}, \theta_{\mp}{}^{b}\}_{\mp}$ are not singular. In these later cases the quantization is postulated to be effected through the rules†

$$\xi_{\mp}\{\,,\,\}_{\mp}{}^{*} \to [\,,\,]_{\mp} \tag{1.1}$$

where $\{\,,\,\}_{\mp}{}^{*}$ are the minus (plus) Dirac brackets. Their explicit forms are:

$$\{F, G\}_{\mp}{}^{*} = \{F, G\}_{\mp} - \{F, \theta_{\mp}{}^{a}\}_{\mp} C^{\mp}_{ab}, \{\theta_{\mp}{}^{b}, G\}_{\mp} \tag{1.2}$$

with

$$C^{\mp}_{ab}\{\theta_{\mp}{}^{b}, \theta_{\mp}{}^{c}\}_{\mp} = \delta_{a}{}^{c} \tag{1.3}$$

In equation (1.1) $\xi_{-} = i$ and ξ_{+} is a parameter whose values have been discussed elsewhere (Kálnay & Ruggeri, 1972). The rules (1.1) are intended to generalize the more usual ones

$$\xi_{\mp}\{\,,\,\}_{\mp} \to [\,,\,]_{\mp} \tag{1.4}$$

valid for unconstrained systems.

In a recent work (Kálnay & Ruggeri, 1972) the authors have considered a classical model which corresponds via the rule (1.1) to the quantum Fermi systems. The quantization of that model has the following interesting peculiarities: (a) the quantum operators algebra is drastically changed by performing a particular gauge transformation,‡ i.e. by going from one Lagrangian to another which differs from the first one by a total time derivative; and (b) the model can also be quantized according to the skew-symmetric (Bose-like) rule and the problems connected with the symmetric case are not present.§ All this suggests that we study the gauge-variance of the minus Dirac bracket under general conditions in order to detect the (eventual) relevance of this variance for the quantization of Bose-like systems. This is, thus, the main purpose of this work.

A closed formula expressing the change of the functional form of $\{\,,\,\}_{-}{}^{*}$ under arbitrary pure gauge transformations is found in Section 3. An important difference between the skew-symmetric and symmetric cases is also stated there. It is shown in Section 4 that, for sufficiently general models, essential changes in the formal structure of the quantized counterparts of Bose-like systems can arise as a result of a gauge transformation. The ultimate reason is that such transformation is not canonical with respect to the minus Dirac bracket. With regard to the symmetric case, it is argued that the gauge-variance of the symmetric quantization rule is also related, in certain cases, to the non-canonicity of an arbitrary gauge transformation with respect to the plus Dirac bracket.

† See Dirac (1964) and Franke & Kálnay (1970).

‡ The name gauge transformation is used through this paper in the sense of Levy–Leblond (1969).

§ Recently Kálnay (1973) has found the same phenomenon when quantizing the free Dirac field.

2. *Notations and Conventions*

We use the name Bose-like model (respectively Fermi-like model) for any classical model which is quantized according to the rule (1.1) with the minus sign (respectively the plus sign).

We consider a system of N degrees of freedom. The coordinates and momenta are collectively denoted by q and p:

$$\begin{aligned} q &= (q_1, q_2, \ldots, q_N) \\ p &= (p_1, p_2, \ldots, p_N) \end{aligned} \tag{2.1}$$

The sum convention is used in any place as well as $\hbar = 1$.

Constraints. N_1 and N_c denote, respectively, the number of primary constraints and the total number of constraints. The constraints themselves, primary or secondary, are denoted by ϕ^n, $n = 1, 2, \ldots, N_1, \ldots, N_c$. They are the same for both the symmetric and the skew-symmetric quantization problem of a given model. Contrarily, the subset $\{\theta_-{}^a\}$, $a = 1, 2, \ldots, N^-{}_\theta$, of minus second-class constraints is usually different from the subset $\{\theta_+{}^a\}$, $a = 1, 2, \ldots, N^+{}_\theta$, of plus second-class constraints.

Brackets. The minus (ordinary, skew-symmetric) or plus (symmetric) brackets are denoted by curly brackets:

$$\{F, G\}_\mp \equiv \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} \mp \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \tag{2.2}$$

Curly brackets with an asterisk: $\{\,,\,\}_\mp{}^*$, denote Dirac brackets. Square brackets are reserved for commutators and anticommutators: $[\,,\,]_\mp$.

Other Notations. The new entities of the theory obtained through a gauge transformation are denoted by a tilde. Examples: $\tilde{p}$ are the new momenta, $\tilde{H}$ the new Hamiltonian, etc. Correspondingly $\widetilde{\{\,,\,\}_-^*}$ is the new minus Dirac bracket.

A partial derivative like $\partial/\partial\tilde{q}_i$ will be written sometimes simply as ∂_i, and

$$\partial = (\partial_1, \partial_2, \ldots, \partial_N) \tag{2.3}$$

3. *Gauge-Variance of the Minus Dirac Bracket*

3.1. *Short Review of the Canonical Formalism for Constrained Systems*

For the following considerations it will be useful to make here a brief review of Dirac's canonical formalism for constrained systems. We shall follow closely Dirac's own work.†

We consider a general classical system whose dynamical properties are derived from a Lagrangian $L(q,\dot{q};t)$. From the defining relations:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{3.1.1}$$

† See, for example, Dirac (1964). See also Franke & Kálnay (1970).

a certain number of relations between q and p sometimes arise. They are called primary constraints and are written as:

$$\phi^n \approx 0, \qquad n = 1, \ldots, N_1 \tag{3.1.2}$$

Because of these constraints the ordinary canonical formalism fails. The canonical equations must now be written in the form:

$$\dot{q}_i = (\partial H/\partial p_i)_q + u_m(\partial \phi^m/\partial p_i)_q \tag{3.1.3a}$$

$$-\dot{p}_i = (\partial H/\partial q_i)_p + u_m(\partial \phi^m/\partial q_i)_p \tag{3.1.3b}$$

where, as in the ordinary theory,

$$H \equiv \dot{q}_i p_i - L \tag{3.1.4}$$

is the Hamiltonian and

$$u_m(q, \dot{q}, \ldots; t), \qquad m = 1, \ldots, N_1 \tag{3.1.5}$$

are non-canonical variables which, together with q and p, are necessary to account for a complete description of the time evolution of the system.

The primary constraints are usually not the only constraints of the problem. Others may appear by eliminating the u's from the consistence equations

$$\dot{\phi}^n = (\partial \phi^n/\partial t) + \{\phi^n, H\}_- + u_m\{\phi^n, \phi^m\}_- \approx 0 \tag{3.1.6}$$

The additional constraints must also satisfy consistence equations like equation (3.1.6). The procedure is iterated until no new constraints appear, the remaining consistence equations serving to restrict the u's. The additional constraints generated this way (called secondary constraints) will also be denoted by ϕ^n but now n runs from $N_1 + 1$ to N_c. As regards the skew-symmetric (symmetric) quantization procedure, this whole set of N_c constraints is conveniently separated in two distinct subsets: the minus (plus) first-class constraints and the minus (plus) second-class constraints. Every minus (plus) first-class constraint has a minus (plus) Poisson bracket with any other constraint that vanishes at least modulo the constraints themselves. This is not true for the second-class constraints which can always be chosen so that the matrices $C^{\mp}$, defined by equation (1.3), exist.†

3.2. *Gauge-Variance*

Let us now consider the changes which occur in the elements of the canonical formalism, sketched above, when the pure gauge transformation

$$q_i \to \tilde{q}_i = q_i \tag{3.2.1a}$$

$$p_i \to \tilde{p}_i = p_i + \partial f/\partial q_i \tag{3.2.1b}$$

† For this we must (eventually) replace the constraints by non-singular linear combinations of them in such a way that we get as many minus (plus) first-class constraints as possible.

is performed. Here f is an arbitrary function of the coordinates and time. Under equations (3.2.1) the Lagrangian changes as

$$L \rightarrow \tilde{L} = L + df/dt \tag{3.2.2}$$

To present in a concise form the changes induced by the gauge transformation it is useful to introduce the associated operator $\mathscr{D}_f$, defined by:

$$(\mathscr{D}_f F)(\tilde{q},\tilde{p};t) \equiv F(\tilde{q},\tilde{p} - \partial f(\tilde{q};t);t) \tag{3.2.3}$$

This operator has the following properties:

$$\text{(i)}\ \mathscr{D}_f \text{ is linear} \tag{3.2.4a}$$
$$\text{(ii)}\ \mathscr{D}_f \mathscr{D}_g = \mathscr{D}_g \mathscr{D}_f = \mathscr{D}_{f+g} \tag{3.2.4b}$$
$$\text{(iii)}\ \mathscr{D}_0 = 1 \tag{3.2.4c}$$
$$\text{(iv)}\ \mathscr{D}_f^{-1} = \mathscr{D}_{-f} \tag{3.2.4d}$$
$$\text{(v)}\ \mathscr{D}_f(F.G) = \mathscr{D}_f(F).\mathscr{D}_f(G) \tag{3.2.4e}$$
$$\text{(vi)}\ (\partial/\partial\tilde{q}_i)(\mathscr{D}_f F) = \mathscr{D}_f(\partial F/\partial\tilde{q}_i) - (\partial^2 f/\partial\tilde{q}_i\,\partial\tilde{q}_j)\,\mathscr{D}_f(\partial F/\partial\tilde{p}_j) \tag{3.2.4f}$$
$$\text{(vii)}\ (\partial/\partial\tilde{p}_i)(\mathscr{D}_f F) = \mathscr{D}_f(\partial F/\partial\tilde{p}_i) \tag{3.2.4g}$$
$$\text{(viii)}\ (\partial/\partial t)(\mathscr{D}_f F) = \mathscr{D}_f(\partial F/\partial t) - (\partial^2 f/\partial t\,\partial\tilde{q}_j)\,\mathscr{D}_f(\partial F/\partial\tilde{p}_j) \tag{3.2.4h}$$

It is not difficult to see that the new momenta, Hamiltonian and primary constraints may now be written as†

$$\tilde{p} = \mathscr{D}_f^{-1} p \tag{3.2.5}$$
$$\tilde{H}(\tilde{q},\tilde{p};t) = (\mathscr{D}_f H)(\tilde{q},\tilde{p};t) - (\partial f/\partial t)(\tilde{q};t) \tag{3.2.6}$$
$$\tilde{\phi}^n(\tilde{q},\tilde{p};t) = (\mathscr{D}_f \phi^n)(\tilde{q},\tilde{p};t) \quad n = 1,\ldots,N_1 \tag{3.2.7}$$

Furthermore, we have also

$$H(q,p;t) = (\mathscr{D}_f H)(\tilde{q},\tilde{p};t) = \tilde{H}(\tilde{q},\tilde{p};t) + (\partial f/\partial t)(\tilde{q};t) \tag{3.2.8}$$
$$\phi^n(q,p;t) = (\mathscr{D}_f \phi^n)(\tilde{q},\tilde{p};t) = \tilde{\phi}^n(\tilde{q},\tilde{p};t), \qquad n = 1,\ldots N_1 \tag{3.2.9}$$

Through these last relations equations (3.1.3) may be rewritten in the form:

$$\dot{\tilde{q}}_i = (\partial\tilde{H}/\partial\tilde{p}_i)_{\tilde{q}} + u_m(\partial\tilde{\phi}^m/\partial\tilde{p}_i)_{\tilde{q}} \tag{3.2.10a}$$
$$-\dot{\tilde{p}}_i = (\partial\tilde{H}/\partial\tilde{q}_i)_{\tilde{p}} + u_m(\partial\tilde{\phi}^m/\partial\tilde{q}_i)_{\tilde{p}} \tag{3.2.10b}$$

which are the new equations of motion. Equations (3.2.10) show that we can take as the new non-canonical variables $\tilde{u}_m$ the same as before:

$$\tilde{u}_m(\tilde{q},\dot{\tilde{q}},\ldots;t) = u_m(q,\dot{q},\ldots;t) \tag{3.2.11}$$

Finally, due to equation (3.2.9) the consistence equations for the new primary constraints are just equations (3.1.6). It follows then that *all* the

† There is some ambiguity in the choice of the functional form of the constraints because linear combinations of the $\tilde{\phi}^n$ may also be chosen as new constraints. We are making the assumption that the same algebraic manipulations achieved when working with the original problem (i.e. that whose Lagrangian is L) are also accomplished after the gauge transformation has been performed. This remark is also important in regard to equations (3.2.12) and (3.2.14) below.

new secondary constraints may be chosen so that they have the form (3.2.7). In summary we have:

$$\tilde{\phi}^n(\tilde{q},\tilde{p};t) = (\mathscr{D}_f \phi^n)(\tilde{q},\tilde{p};t), \qquad n = 1,\ldots,N_c \tag{3.2.12}$$

The character of the new constraints will be considered now. For this, let us note that by virtue of equations (3.2.4f) and (3.2.4g) the following relation holds for any F and G:

$$\mathscr{D}_f\{F,G\}_- = \{\mathscr{D}_f F, \mathscr{D}_f G\}_- \tag{3.2.13}$$

This relation expresses the well-known fact that a gauge transformation is canonical with respect to the minus Poisson bracket. A moment of reflection then shows that if ϕ^n is minus first (second) class then $\tilde{\phi}^n$ is minus (second) class. Then

$$\tilde{\theta}_-{}^a(\tilde{q},\tilde{p};t) = (\mathscr{D}_f \theta_-{}^a)(\tilde{q},\tilde{p};t), \qquad a = 1,\ldots,N^-{}_\theta \tag{3.2.14}$$

and also, by equation (1.3),

$$\tilde{C}^-_{ab}(\tilde{q},\tilde{p};t) = (\mathscr{D}_f C^-_{ab})(\tilde{q},\tilde{p};t) \tag{3.2.15}$$

At this point an important difference between the symmetric and the skew-symmetric cases arise. This difference lies in the fact that a relation like (3.2.13) is not valid in general for the plus Poisson bracket because a gauge transformation is not generally canonical with respect to this bracket. Thus, in a general problem, not only is there a change in the functional form of the constraints under a gauge transformation but also the subset of irreducible plus second-class constraints may be altered. This fact strongly suggests that in most cases there will be a radical change of the plus Dirac bracket relations (and consequently of the anticommutator algebra derived from them through the rule (1.1)) after performing a gauge transformation.†

The validity of relation (3.2.13) permits one to obtain a simple closed formula for the gauge-variance of the minus Dirac bracket. The same seems difficult to find in general for the plus Dirac bracket. To find that formula, note that the new Dirac bracket is given by :

$$\{F,G\}_-^{\tilde{*}} \equiv \{F,G\}_- - \{F,\tilde{\theta}_-{}^a\}_- \tilde{C}^-_{ab}\{\tilde{\theta}_-{}^b, G\}_- \tag{3.2.16}$$

Applying now $\mathscr{D}_f^{-1}$ to both sides of (3.2.16) and using equations (3.2.4a), (3.2.4d), (3.2.4e), (3.2.13), (3.2.14) and (3.2.15) we arrive at:

$$\mathscr{D}_f^{-1}\{F,G\}_-^{\tilde{*}} = \{\mathscr{D}_f^{-1}F, \mathscr{D}_f^{-1}G\}_- - \{\mathscr{D}_f^{-1}F, \theta_-{}^a\}_- C^-_{ab}\{\theta_-{}^b, \mathscr{D}_f^{-1}G\}_- \tag{3.2.17}$$

or, in a concise form,

$$\{F,G\}_-^{\tilde{*}} = \mathscr{D}_f\{\mathscr{D}_f^{-1}F, \mathscr{D}_f^{-1}G\}_-^* \tag{3.2.18}$$

This formula is the general relation we looked for. It expresses the gauge-variance of the functional form of the minus Dirac bracket.‡

† This is just the kind of problem we have found in two previous works where classical analogues of Fermi systems were considered. See Kálnay & Ruggeri (1972) and Kálnay (1973).

‡ A formula similar to (3.2.18) may be found for the variance of the Dirac bracket under any transformation which is canonical with respect to the minus Poisson bracket.

4. *Quantization*

Let us now comment on the implications of relation (3.2.18) for the quantization rule of constrained Bose-like systems. As is known, the rule (1.1), as well as the ordinary quantization rule (1.4), is consistent if it is used only to find the commutation relations obeyed by the coordinates and momenta of the quantized system. These operators correspond to the canonical variables of the classical system, thus we only need to consider here their minus brackets.

Let us call

$$\omega_1 = q_1, \ldots, \omega_N = q_N, \qquad \omega_{N+1} = p_1, \ldots, \omega_{2N} = p_N, \qquad \omega = (\omega_1, \ldots, \omega_{2N}) \tag{4.1}$$

and denote

$$K_{\alpha\beta}(\omega) = \{\omega_\alpha, \omega_\beta\}_-^*(\omega) \tag{4.2a}$$

$$\tilde{K}_{\alpha\beta}(\tilde{\omega}) = \{\tilde{\omega}_\alpha, \tilde{\omega}_\beta\}_-^{\tilde{*}}(\tilde{\omega}) \tag{4.2b}$$

Let $\mathcal{O}$ be an ordering prescription used when quantizing:

$$K_{\alpha\beta}(\omega) \to \mathcal{O}K_{\alpha\beta}(\omega_{\mathrm{op}}) \tag{4.3a}$$

$$\tilde{K}_{\alpha\beta}(\tilde{\omega}) \to \mathcal{O}\tilde{K}_{\alpha\beta}(\tilde{\omega}_{\mathrm{op}}) \tag{4.3b}$$

(The quantization must be done within the same theory in order to compare the quantization of the systems whose Lagrangians are L and $\tilde{L}$, so that $\mathcal{O}$ must be the same for both.)

The quantization rule for a system governed by L is

$$[\omega_{\alpha,\mathrm{op}}, \omega_{\beta,\mathrm{op}}]_- = i\mathcal{O}K_{\alpha\beta}(\omega_{\mathrm{op}}) \tag{4.4a}$$

and the one for the system governed by $\tilde{L}$ is

$$[\tilde{\omega}_{\alpha,\mathrm{op}}, \tilde{\omega}_{\beta,\mathrm{op}}]_- = i\mathcal{O}\tilde{K}_{\alpha\beta}(\tilde{\omega}_{\mathrm{op}}) \tag{4.4b}$$

Usually two systems whose Lagrangians differ by a time derivative, as in equation (3.2.2), are considered as physically equivalent. In order to be able to question this assumption we shall temporarily admit it. We shall look for the consequences of the following *working hypothesis*: the quantum rules (4.4a) and (4.4b) stand for the same physical system. (We do not assume that they are written within the same representation.) The phase-space frame whose coordinates are represented by ω is then equivalent to the frame whose coordinates are $\tilde{\omega}$, so that an unitary transformation U_{op} exists such that:

$$\tilde{\omega}_{\mathrm{op}} = U_{\mathrm{op}}\,\omega_{\mathrm{op}}\,U_{\mathrm{op}}^{-1} \tag{4.5}$$

It follows then from equation (4.4b) that

$$[\omega_{\alpha,\mathrm{op}}, \omega_{\beta,\mathrm{op}}]_- = i\mathcal{O}\tilde{K}_{\alpha\beta}(\omega_{\mathrm{op}}) \tag{4.6}$$

Comparing equation (4.6) with equation (4.4a) and going to the classical limit (where $\mathcal{O}$ becomes irrelevant) we are led to

$$\tilde{K}_{\alpha\beta}(z) = K_{\alpha\beta}(z) \qquad \text{for all } z \text{ and } \alpha, \beta = 1, \ldots, 2N \tag{4.7}$$

On the other hand, the function $\tilde{K}$, whose form is given by equation (4.2b), can be taken at any variables whatever. Choosing the ω variables we can write

$$\tilde{K}_{\alpha\beta}(\omega) = \{\omega_\alpha, \omega_\beta\}_-^{\tilde{*}}(\omega) \tag{4.8}$$

and then from equations (4.7), (4.2a) and (4.8) we have

$$\{\omega_\alpha, \omega_\beta\}_-^{\tilde{*}}(\omega) = \{\omega_\alpha, \omega_\beta\}_-^*(\omega) \tag{4.9}$$

We can now take into account equation (3.2.18) *written in the ω variables*, for $F = \omega_\alpha$ and $G = \omega_\beta$ to find

$$\mathscr{D}_f\{\mathscr{D}_f^{-1}\,\omega_\alpha, \mathscr{D}_f^{-1}\,\omega_\beta\}_-^* = \{\omega_\alpha, \omega_\beta\}_-^* \tag{4.10a}$$

or what is the same

$$\mathscr{D}_f^{-1}\{\omega_\alpha, \omega_\beta\}_-^* = \{\mathscr{D}_f^{-1}\,\omega_\alpha, \mathscr{D}_f^{-1}\,\omega_\beta\}_-^* \tag{4.10b}$$

We then see, as could be expected, that in order that the gauge transformation induce a mere change of representation at the quantum level, it must be canonical with respect to the minus Dirac bracket. To convince ourselves that this is not true in general let us consider an infinitesimal gauge transformation: $f(q;t) = \varepsilon\not{f}(q;t)$ where ε is an infinitesimal parameter. From equation (3.2.3) we find

$$\mathscr{D}_{\mp f} = 1 \pm \varepsilon(\partial_k \not{f})\frac{\partial}{\partial p_k} + O(\varepsilon^2) \tag{4.11}$$

which, when replaced in equation (4.10b), leads us to:

$$\partial_k \not{f}\,\partial\{\omega_\alpha, \omega_\beta\}_-^*/\partial p_k = \{(\partial_k \not{f})(\partial\omega_\alpha/\partial p_k), \omega_\beta\}_-^* + \{\omega_\alpha, (\partial_k \not{f})\,\partial\omega_\beta/\partial p_k\}_-^* \tag{4.12}$$

or, in explicit form,

$$(\partial_k \not{f})(\partial/\partial p_k)\{q_i, q_j\}_-^* = 0 \tag{4.13a}$$

$$(\partial_k \not{f})(\partial/\partial p_k)\{q_i, p_j\}_-^* = \{q_i, \partial_j \not{f}\}_-^* \tag{4.13b}$$

$$(\partial_k \not{f})(\partial/\partial p_k)\{p_i, p_j\}_-^* = \{\partial_i \not{f}, p_j\}_-^* + \{p_i, \partial_j \not{f}\}_-^* \tag{4.13c}$$

But relations (4.13) are not generally satisfied, as can be easily shown by considering examples as the one worked out below. *We are then led to conclude that if Dirac's quantization rule* (*upper sign of* (1.1)) *for constrained systems are right, then the previous working hypothesis was wrong so that: two quantum systems whose classical Lagrangians differ by a time derivative may not be physically equivalent.*

Example. As a simple (and rather academic) example of the general failure of (4.13) consider the two-dimensional system whose Lagrangian is:

$$L = (1/2)(\lambda^{-2}\dot{q}_1{}^2 - q_1{}^2)\,q_2 \tag{4.14}$$

$\lambda \neq 0$ is a parameter. The evolution equations derived from (4.14) have the solution (where $\sigma = \pm 1$)†

$$q_1(t) = q_1^\sigma(0) \exp(\sigma\lambda t) \tag{4.15a}$$

$$q_2(t) = q_2^\sigma(0) \exp(-2\sigma\lambda t) \tag{4.15b}$$

$$p_1(t) = \sigma\lambda^{-1} q_1^\sigma(0) q_2^\sigma(0) \exp(-\sigma\lambda t) \tag{4.15c}$$

$$p_2(t) = 0 \tag{4.15d}$$

It is elementary to show that in this example there are only two constraints:

$$\theta^1 = p_2 \approx 0 \tag{4.16a}$$

and

$$\theta^2 = (\lambda^2 p_1{}^2/q_2{}^2) - q_1{}^2 \approx 0 \tag{4.16b}$$

They give rise to Dirac bracket relations as e.g.

$$\{q_1, q_2\}_-{}^* = (q_2/p_1) \tag{4.17}$$

From this last relation we see that any f that depends on q_1 violates condition (4.13a) so that it is proved that the quantization of this model system is altered by most gauge transformations. Moreover, in this simple case it is seen directly from equation (4.17) that adding to the Lagrangian the particular time derivative $q_1 \dot{q}_1$ the value of $\{q_1, q_2\}_-{}^*$ changes. We remark that this happens in spite of the fact that neither q_1 nor q_2 are changed by pure gauge transformations.

5. *Summary*

We have devoted this work to analyze the gauge-variance of the minus Dirac bracket and its relevance to the quantization rule for Bose-like systems. We have seen that the variance is entirely due to the change in the functional form of the constraints. For Fermi-like systems the situation is probably worse because the subset of second-class constraints may also be altered.

As a result of this work the formal structure of a quantized system may be altered by performing a gauge transformation, because these transformations are not generally canonical with respect to the Dirac brackets. As these brackets are involved in the quantization rules for constrained systems we have the following alternative: (a) as customarily implied the change of the Lagrangian by a total time derivative is to be considered as devoid of any physical significance, consequently the quantization rule for sufficiently generally constrained systems must be modified; or (b) the quantization scheme holds good but a physical difference may exist between two con-

† We suppose that the q's are non-identically zero functions of the time with continuous first derivatives.

strained systems whose classical Lagrangians L and $\tilde{L}$ are related by a gauge transformation:

$$\tilde{L} = L + df/dt$$

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